

Self-avoiding walks on scale-free networks

Carlos P. Herrero

Instituto de Ciencia de Materiales, Consejo Superior de Investigaciones Científicas (CSIC), Campus de Cantoblanco, 28049 Madrid, Spain

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Several kinds of walks on complex networks are currently used to analyze search and navigation in different systems. Many analytical and computational results are known for random walks on such networks. Self-avoiding walks (SAW's) are expected to be more suitable than unrestricted random walks to explore various kinds of real-life networks. Here we study long-range properties of random SAW's on scale-free networks, characterized by a degree distribution $P(k) \sim k^{-\gamma}$. In the limit of large networks (system size $N \rightarrow \infty$), the average number s_n of SAW's starting from a generic site increases as μ^n , with $\mu = \langle k^2 \rangle / \langle k \rangle - 1$. For finite N , s_n is reduced due to the presence of loops in the network, which causes the emergence of attrition of the paths. For kinetic growth walks, the average maximum length $\langle L \rangle$ increases as a power of the system size: $\langle L \rangle \sim N^\alpha$, with an exponent α increasing as the parameter γ is raised. We discuss the dependence of α on the minimum allowed degree in the network. A similar power-law dependence is found for the mean self-intersection length of nonreversal random walks. Simulation results support our approximate analytical calculations.

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I. INTRODUCTION

Many natural and artificial systems have a network structure, where nodes represent typical system units and edges represent interactions between connected pairs of units. Thus, complex networks are currently used to model several kinds of real-life systems (social, biological, technological, economic) and to study different processes taking place on them [1–3]. In recent years, new models of complex networks have been designed to explain empirical data in several fields. This is the case of the so-called small-world [4] and scale-free networks [5], which incorporate various aspects of real systems. These complex networks provide us with the underlying topological structure to analyze processes such as spread of infections [6,7], signal propagation^{4,8}, and random spreading of information [9,10]. They have been also employed to study statistical physical problems as percolation [6,11] and cooperative phenomena [12,13].

In a scale-free (SF) network the degree distribution $P(k)$, where k is the number of links connected to a node, has a power-law decay $P(k) \sim k^{-\gamma}$. This kind of networks have been found in social systems [14], for protein interactions [15], in the internet [16], and in the World Wide Web [17]. In both natural and artificial networks, the exponent γ controlling the degree distribution is usually in the range $2 < \gamma < 3$ [3,18]. The origin of such power-law degree distributions was addressed by Barabási and Albert [5], who argued that two ingredients are sufficient to explain the scale-free character of many real-life networks, namely: growth and preferential attachment. They found that the combination of both criteria yields nonequilibrium SF networks with an exponent $\gamma=3$. One can also study equilibrium SF networks, defined as statistical ensembles of random networks with a given degree distribution $P(k) \sim k^{-\gamma}$ [3], for which one can analyze several properties as a function of the exponent γ . SF networks display the so-called small-world effect, and they have

been found to be ultrasmall, in the sense that the mean distance between sites increases with the network size N slower than $\log N$ [19].

Social networks form the substrate where dynamical processes such as disease propagation and information spreading take place. These networks have the property of being searchable—i.e., people (nodes in a network) can direct messages through their network of acquaintances to reach a distant specific target in only a few steps [20–22]. It is clear that the structure of such networks will play an important role in these dynamical processes, which are usually studied by means of stochastic dynamics and random walks. Several characteristics of random walks on complex networks have been analyzed in connection with diffusion and exploration processes [23–25]. In this context, it is known that some processes, such as navigation and exploratory behavior are neither purely random nor totally deterministic [26] and can be described by walks on graphs [27,28].

Self-avoiding walks (SAW's) can be more effective than unrestricted random walks in exploring a network, since they cannot return to sites already visited. This property has been used by Adamic *et al.* [28] to define local search strategies in scale-free networks. However, the self-avoiding property causes attrition of the paths, in the sense that a large fraction of paths generated in a stochastic manner have to be abandoned because they are overlapping. This can be a serious limitation to explore networks with pure SAW's.

SAW's have been traditionally used to model structural and dynamical properties of macromolecules [29,30]. They are also useful to characterize complex crystal structures [31] and to study critical phenomena in lattice models [32]. Universal constants for SAW's on regular lattices have been discussed by Privman *et al.* [33]. In our context of complex networks, the asymptotic properties of SAW's have been studied recently in small-world networks [34].

Here we study long-range properties of SAW's on equilibrium scale-free networks and discuss the “attrition prob-

lem.” The number of surviving walks to a given length n is obtained by an approximate analytical procedure, and the results are compared with those obtained from numerical simulations. In particular, we find that the number of surviving walks after n steps scales as a power of the system size N . We note that the term “length” is employed throughout this paper to indicate the (dimensionless) number of steps of a walk, as usually done in the literature on networks [3].

The paper is organized as follows. In Sec. II we give some definitions and concepts related to SAW’s, along with details of our computational method. In Sec. III we calculate the number and end-to-end separation of SAW’s in (uncorrelated) scale-free networks. In Sec. IV we analyze the length at which nonreversal random walks intersect themselves in these networks (self-intersection length), and in Sec. V we calculate the average attrition length of kinetic growth SAW’s, at which they cannot continue without violating the self-avoidance condition. The paper closes with some conclusions in Sec. VI.

II. BASIC DEFINITIONS AND METHOD

A self-avoiding walk is defined as a walk along the bonds of a given network which can never intersect itself. The walk is restricted to moving to a nearest-neighbor site during each step, and the self-avoiding condition constrains the walk to occupy only sites which have not been previously visited in the same walk.

The simplest procedure to obtain SAW’s consists just in generating ordinary random walks and stop when they arrive at a node already visited. A problem with this sampling algorithm in regular lattices is the exponentially rapid attrition for long walks, since the probability of an n -step walk being self-avoiding behaves for large n as $e^{-\lambda n}$, where λ is the so-called attrition constant [35]. Due to this limitation, more sophisticated schemes based on Monte Carlo sampling, have been employed to generate SAW’s with the correct weight and to obtain ensemble averages of several quantities [35]. This has allowed, for example, the modeling of the equilibrium statistics of linear polymers in dilute solutions. In general, for networks including nodes with different degrees (contrary to usual regular lattices), sampling by using simple random walks introduces a bias in the weight of different SAW’s.

One can also consider kinetically grown SAW’s, which can be more adequate to analyze dynamic processes. Such walks are well suited to study, for example, search or navigation processes on networks, where they are assumed to grow step by step in a temporal sequence. In the following we will consider two kinds of growing walks. The first kind will be “nonreversal” self-avoiding walks [35]. In these walks one randomly chooses the next step from among the neighboring nodes, excluding the previous one. If it happens that one chooses an already visited node, then the walk stops (see Fig. 1). These walks will allow us to study the “self-intersection length” (see Sec. IV). The second kind of walks considered here are kinetic growth walks [36], in which one randomly chooses the next step among the neighboring unvisited sites and stops growing when none are available.

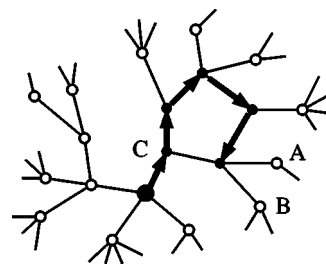


FIG. 1. Schematic diagram showing a nonreversal random walk of length $n=5$ on the realization of a random graph. Open and solid circles represent unvisited and visited nodes, respectively. The starting node is indicated by a larger circle. The nonreversal condition allows in principle for the next (sixth) step three possible nodes (denoted A , B , and C). The self-avoiding condition excludes node C for the sixth step. For a nonreversal SAW one chooses among nodes A , B , and C . If C is selected, then the walk stops. For a kinetic growth walk, one chooses A or B .

These walks were studied to describe the irreversible growth of linear polymers [36] and will allow us to consider the “attrition length” for a walk on a given network (see Sec. V). Note that kinetic growth walks are less sensitive to attrition than nonreversal SAW’s, since in the former the walker always escapes whenever a way exists.

We consider SF networks with degree distribution $P(k) \sim k^{-\gamma}$. They are characterized, apart from the exponent γ and the system size N , by the minimum degree k_0 , which affects markedly some characteristics of SAW’s in these networks (see below). We assume that $P(k)=0$ for $k < k_0$. Our networks are uncorrelated, in the sense that degrees of nearest neighbors are statistically independent. This means that the joint probability $P(k, k')$ fulfills the relation [3]

$$P(k, k') = \frac{kk'}{\langle k \rangle^2} P(k)P(k'). \quad (1)$$

For the numerical simulations we have generated networks with several values of γ , k_0 , and N . To generate a network, once we define the number of nodes, N_k , with degree k , we ascribe a degree to each node according to the set $\{N_k\}$ and then connect at random ends of links (giving a total of $L = \sum_k k N_k / 2$ connections) with the conditions that (i) no two nodes can have more than one bond connecting them and (ii) no node can be connected by a link to itself. We have checked that networks generated in this way are uncorrelated; i.e., they fulfill Eq. (1). All networks considered here contain a single component; i.e., any node in a network can be reached from any other node by traveling through a finite number of links. For each set of parameters (γ, k_0, N) , we considered different network realizations, and for a given network we selected at random the starting nodes for the SAW’s. For each considered parameter set, the total number of generated SAW’s amounted to about 5×10^5 .

For regular lattices, the number s_n of different SAW’s starting from a generic site has an asymptotic dependence for large n [33]: $s_n \sim n^{\Gamma-1} \mu^n$, where Γ is a critical exponent which depends on the lattice dimension and μ is the so-called “connective constant” or effective coordination num-

ber of the considered lattice [37]. In general, for a lattice with connectivity k_0 , one has $\mu \leq k_0 - 1$. This parameter μ can be obtained as the limit

$$\mu = \lim_{n \rightarrow \infty} \frac{s_n}{s_{n-1}}. \quad (2)$$

The connective constant depends upon the particular topology of each lattice and has been determined very accurately for two- and three-dimensional lattices [35].

For Poissonian and scale-free networks the number of SAW's of length n depends on the considered starting node of the network. In the sequel we will call s_n the average number of SAW's of length n —i.e., the mean value obtained (for each n) by averaging over the network sites and over different network realizations (for given γ , k_0 , and N). For Erdős-Rényi random networks with Poissonian distribution of degrees, one has $s_n^{rd} = \langle k \rangle^n$ [34], and therefore the connective constant is $\mu = \langle k \rangle$. In connection with this, we note that for a Bethe lattice (or Cayley tree) with connectivity k_0 , the number of SAW's is given by $s_n^{BL} = k_0(k_0 - 1)^{n-1}$, and one has $\mu_{BL} = k_0 - 1$.

III. GENERAL CHARACTERISTICS OF SAW'S

A. Number of walks

We calculate first the average number, r_n , of different n -step unrestricted walks starting from a node chosen at random. One trivially has $r_1 = \langle k \rangle$. To calculate r_n for $n > 1$, one needs the degree distribution for nodes at which one arrives following a random edge. Thus, given a generic node and a link starting on it, we call $Q(k)$ the degree distribution for the other end of the link. The probability of reaching a node with connectivity k is proportional to k ; therefore,

$$Q(k) = \frac{k}{\langle k \rangle} P(k), \quad (3)$$

where $\langle k \rangle$ in the denominator is a normalization factor. Then, the average number of two-step random walks is given by $r_2 = r_1 \langle k \rangle_Q$, where the subscript Q indicates that the average value is taken with the probability distribution $Q(k)$. We find $r_2 = \langle k^2 \rangle$ [average values without subscripts are taken with the degree distribution $P(k)$]. For $n > 2$ we have $r_n = r_{n-1} \langle k \rangle_Q$ and then

$$r_n = \langle k \rangle \left(\frac{\langle k^2 \rangle}{\langle k \rangle} \right)^{n-1}. \quad (4)$$

We have checked that this expression for r_n and those given below for SAW's coincide with those derived by using a generating function for the degree distribution [28,38].

We now calculate the average number s_n of different self-avoiding walks of n steps starting from a node taken at random in uncorrelated networks. We will first consider the case $n/N \rightarrow 0$ (thermodynamic limit). For $n=1$, one has $s_1 = \langle k \rangle$. For $n > 1$ we take into account that each $(n-1)$ -step walk arriving at a node with degree k gives rise to $k-1$ n -step walks. Thus, we have $s_n = s_{n-1} \langle k-1 \rangle_Q$, which yields

$$s_n = \langle k \rangle \left(\frac{\langle k^2 \rangle}{\langle k \rangle} - 1 \right)^{n-1}. \quad (5)$$

Then, the connective constant μ_∞ for $N \rightarrow \infty$ is given by

$$\mu_\infty = \frac{\langle k^2 \rangle}{\langle k \rangle} - 1. \quad (6)$$

This is consistent with the fact that uncorrelated networks are locally tree like and the ratio $\langle k^2 \rangle / \langle k \rangle$ is the average degree of a randomly chosen end node of a randomly chosen link [3]. For SF networks with $\gamma \leq 3$, μ diverges as $N \rightarrow \infty$, as a consequence of the divergence of $\langle k^2 \rangle$. For $\gamma > 3$, we can approximate the average values in Eq. (6) by replacing sums by integrals and find

$$\mu_\infty \approx k_0 \frac{\gamma - 2}{\gamma - 3} - 1. \quad (7)$$

Note that the ratio s_n / s_{n-1} does not depend on n for system size $N \rightarrow \infty$. This is equivalent to assume that nodes in different steps of a nonreversal random walk are different. This means, in other words, that the probability of finding loops with $n' \leq n$ in an n -step walk is negligible. For finite networks, however, there will appear loops of any size [39], introducing corrections to the number of SAW's, and s_n will be lower than given by Eq. (5). These corrections will be of order n/N for $n/N \ll 1$. The effects of this reduction in the number of random SAW's in scale-free networks will be considered in Secs. IV and V.

As indicated above, the number of SAW's on regular lattices scales for large n as $s_n \sim n^{\Gamma-1} \mu^n$, where Γ is a critical exponent which depends on the lattice dimension D , and one has $\Gamma = 1$ for $D > 4$ [33,35]. For the SF networks studied here we find $s_n \sim \mu_\infty^n$, indicating that $\Gamma = 1$, the same exponent as for regular lattices in many dimensions.

B. End-to-end separation

For walks on regular lattices, one usually considers an end-to-end Euclidean distance. Our SF networks, however, lack a metric and a true distance is not defined. Thus, we will consider the end-to-end separation for SAW's on SF networks as a function of the walk length n , the separation between two nodes being the number of links along the shortest path connecting them. In Fig. 2 we present the average separation $\langle d_n \rangle$ from the n th node in a SAW to the starting one ($n=0$) for SF networks with $\gamma=3$, $k_0=3$, and several sizes N . This average separation $\langle d_n \rangle$ increases first linearly with n and finally saturates to a finite value that depends on the system size. This is logical if one takes into account that these networks are locally tree like and for the first steps the minimum separation between nodes n and 0 is $d_n = n$. As n increases, there appear shorter ways connecting nodes n and 0, and finally $\langle d_n \rangle$ becomes independent of n . This saturation for relatively small values of n is consistent with the small average separation between pairs of nodes (diameter) in this kind of networks. In fact, it is known that the diameter increases slowly as $\ln N / \ln \ln N$ for $\gamma=3$ and even more slowly as $\ln \ln N$ for $\gamma < 3$ [19]. The limit of $\langle d_n \rangle$

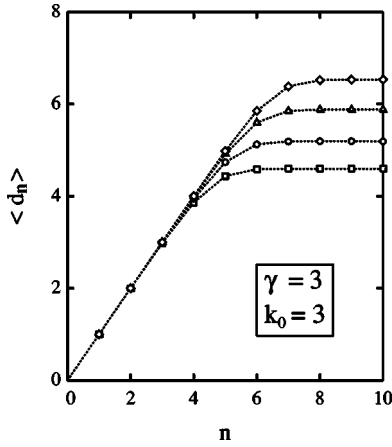


FIG. 2. Average separation between the n th node and the starting one, for kinetically grown SAW's on scale-free networks with $\gamma=3$ and $k_0=3$. Symbols correspond to different system sizes. From bottom to top: $N=2.1 \times 10^3$, 9.6×10^3 , 7.7×10^4 and 6.2×10^5 . Dotted lines are guides to the eye.

for large n in our SAW's is lower than the diameter of the considered networks, because nodes with larger k have a higher probability of being visited than those with smaller k , as indicated above. Since the average separation of a node to all other nodes in a network decreases for increasing degree of the selected node, the average separation $\langle d_n \rangle$ for large n in SAW's is lower than the diameter of the network. For example, for two of the system sizes represented in Fig. 2 ($N=2.1 \times 10^3$ and 7.7×10^4) we have, for large n , $\langle d_n \rangle = 4.60$ and 5.88 versus the diameters of 4.77 and 6.07 , respectively.

For regular lattices, the mean squared end-to-end distance of SAW's scales for large n as $n^{2\nu}$, ν being a dimension-dependent critical exponent. The upper critical dimension for these walks is $D=4$ [40], which means that above this dimension one has $\nu=\frac{1}{2}$, as for Brownian motion (Markovian random walks). On the other hand, for the SF networks studied here, the mean-squared end-to-end separation scales as n^2 in the thermodynamic limit—i.e., with an exponent $\nu=1$. This exponent coincides with that corresponding to $D=1$, reflecting the fact that loops become irrelevant in the considered networks as $N \rightarrow \infty$ (networks become tree-like). This behavior is, however, not captured by SAW's on finite SF networks, for which $\langle d_n^2 \rangle$ converges to a constant, and therefore gives a null exponent for sufficiently large n .

IV. SELF-INTERSECTION LENGTH

As indicated above, finite-size effects on SAW's on any (finite) network will be appreciable as soon as the walks are long enough, as a consequence of the presence of loops in the network. Thus, SAW's are a suitable tool to probe the large-scale topological structure of complex networks. In particular, the probability of a walk intersecting itself will depend on the system size, as well as on the topology of the network under consideration. To study this probability, we consider here nonreversal self-avoiding walks [35], which

stop when they try to visit a node already visited in the same walk. The number of steps of a given walk before intersecting itself will be called the “self-intersection length” of the walk and will be denoted l .

To obtain the mean self-intersection length $\langle l \rangle$ of these walks, we will calculate the probability that a walk stops at step n ($\ll N$). Let us consider for the moment nodes with a given degree k . The average number of nodes with degree k visited after n steps is

$$V_k = nQ(k), \quad (8)$$

and the average number of those yet unvisited is

$$U_k = N_k - V_k = NP(k) - nQ(k). \quad (9)$$

Then, the probability of reaching in step n an unvisited node is $u_k \propto kU_k$ and that of finding one already visited is $v_k \propto (k-2)V_k$. This is due to the fact that a visited node has $k-2$ possible links to reach it, as two of its connections are not available because they were employed earlier: one for an incoming step and one for an outgoing step. Therefore, the probability p_n of finding in step n a visited node with any degree is

$$p_n = \frac{\sum_k v_k}{\sum_k (v_k + u_k)}. \quad (10)$$

Inserting into Eq. (10) expressions (8) and (9) for V_k and U_k and keeping terms linear in n/N , one has

$$p_n \approx \frac{n}{\langle k \rangle N} \sum_k (k-2)Q(k), \quad (11)$$

and, finally,

$$p_n \approx w \frac{n}{N}, \quad (12)$$

where

$$w = \frac{\langle k^2 \rangle - 2\langle k \rangle}{\langle k \rangle^2}. \quad (13)$$

Note that for the networks considered here $w > 0$. In fact, $\langle k^2 \rangle - 2\langle k \rangle > 0$ is the condition to have a giant component in a network [41].

To calculate the probability distribution for the self-intersection length l , we consider M_0 random walks starting from nodes taken at random. We call $M_1(n)$ the number of nonreversal SAW's that remain after n steps (i.e., those which did not find any node visited earlier). Thus, $M_1(n) - M_1(n+1) = p_n M_1(n)$, and considering n as a continuous variable x , we have a differential equation for $M_1(x)$:

$$\frac{1}{M_1} \frac{dM_1}{dx} = -w \frac{x}{N}, \quad (14)$$

which yields, for integer n

$$M_1(n) = M_0 \exp\left(-\frac{w n^2}{2N}\right). \quad (15)$$

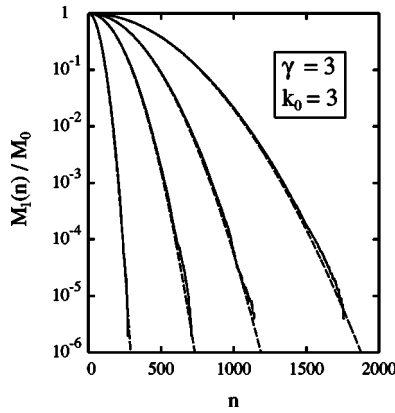


FIG. 3. Fraction of nonreversal SAW's that survive after n steps, without intersecting themselves. Results are plotted for SF networks with $\gamma=3$, $k_0=3$, and several system sizes N . From left to right: $N=3.3 \times 10^3$, 2.6×10^4 , 7.7×10^4 , and 2.1×10^5 . Solid and dashed lines indicate results of numerical simulations and analytical calculations, respectively.

In Fig. 3 we present results for the fraction of surviving walks $M_1(n)/M_0$ for SF networks with $\gamma=3$. We compare the curves derived from Eq. (15) (dashed lines) with those obtained from numerical simulations for networks with $k_0=3$ (solid lines). Both sets of results agree with one another in the sensitivity region of our numerical procedure (down to $\sim 10^{-6}$). For larger n [and lower $M_1(n)/M_0$] one expects terms of higher order in n/N to become relevant and Eq. (15) to be less reliable.

The average self-intersection length of these walks can be obtained as $\langle l \rangle$, with the probability distribution

$$R(l) = w \frac{l}{N} \exp\left(-\frac{wl^2}{2N}\right), \quad (16)$$

which gives the probability of returning to a visited site in step l . Treating l as a continuous variable, we replace sums by integrals and find

$$\langle l \rangle \approx \sqrt{\frac{\pi N}{2w}}. \quad (17)$$

For large N and $\gamma > 2$, $\langle k \rangle$ converges to a finite value and $w \sim \langle k^2 \rangle$; therefore, the mean self-intersection length scales as $\langle l \rangle \sim (N/\langle k^2 \rangle)^{1/2}$. For $\gamma > 3$, $\langle k^2 \rangle$ does not diverge for large N and then $\langle l \rangle \sim \sqrt{N}$. Mean self-intersection lengths derived from Eq. (17) are shown in Fig. 4, along with those found in numerical simulations for several exponents γ . Both methods give results agreeing with one another within the error bars of the numerical simulations. In general, we find $\langle l \rangle \sim N^\beta$, with an exponent β that decreases from 0.5 to 0.25 as γ decreases from 3 to 2. From the distribution for l given by Eq. (16) one finds a mean-square deviation for the self-intersection length of the walks: $\sigma_l^2 = CN/w$, with a constant $C = 2 - \pi/2$. This means that $\sigma_l/\langle l \rangle \approx 0.52$.

We note that the probability distribution $R(l)$ (and therefore the average value $\langle l \rangle$) is independent of the minimum degree k_0 . It depends, apart from the system size, on the

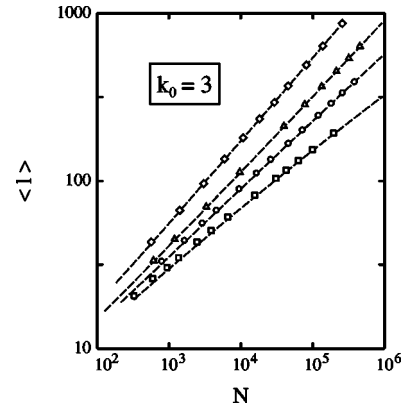


FIG. 4. Mean self-intersection length $\langle l \rangle$ as a function of system size, for SF networks with $k_0=3$ and several values of the exponent γ . From top to bottom: $\gamma=5$, 3, 2.5, and 2. Symbols are results of numerical simulations and dashed lines were obtained from Eq. (17). Error bars of simulation results are less than the symbol size.

exponent γ of the degree distribution through the mean values $\langle k \rangle$ and $\langle k^2 \rangle$. In Fig. 5 we present the mean self-intersection length for SF networks with $\gamma=2.5$ and several values of k_0 , as derived from numerical simulations. Different k_0 values give a unique N dependence of $\langle l \rangle$. Thus, the dependence of $\langle l \rangle$ on k_0 , which should appear in Eq. (17) through the dependence of w on k_0 , is negligible for our purposes. However, the minimum degree k_0 affects strongly other properties of SAW's, such as the attrition length studied in the following section.

V. ATTRITION LENGTH

In this section we consider random SAW's, which travel on the network until they arrive at a node (called hereafter the "blocking node"), where they cannot continue because all adjacent nodes have been already visited and are not available for the walk. These are kinetic growth walks, as defined by Majid *et al.* [36]. The number of steps of a given walk until being blocked will be called the attrition length of the walk and will be denoted L .

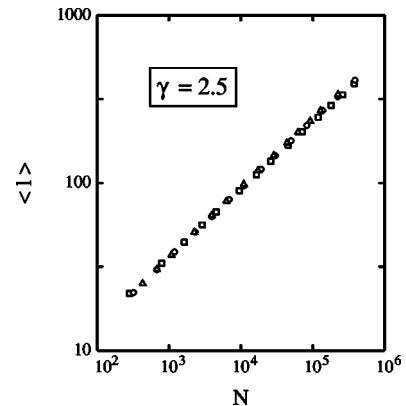


FIG. 5. Mean self-intersection length $\langle l \rangle$ as a function of system size for SF networks with $\gamma=2.5$ and several values of the minimum degree k_0 : squares, $k_0=3$; circles, $k_0=5$; triangles, $k_0=7$.

We now calculate the average attrition length of kinetic growth walks and obtain its asymptotic dependence for large system size N . To find this average length we will derive a probability distribution for L in a way similar to that employed above for the self-intersection length. With this purpose we note that a blocking node for a kinetic growth walk is characterized by the fact that all its links except one (employed for an incoming step) connect it with nodes previously visited. Then, for step n of a walk and for a given degree k , the average number N'_k of blocking nodes present in the network is given by the binomial formula

$$N'_k = kN_k p_n^{k-1} (1 - p_n) \approx kN_k p_n^{k-1}, \quad (18)$$

where p_n ($\ll 1$ for $n \ll N$) is the average fraction of links joining a generic node with nodes visited earlier, as given in Eq. (12). This means that $N'_k \sim k^{1-\gamma} p_n^{k-1}$, and then one has $N'_k \ll N'_{k_0}$ for $k > k_0$. Thus, the average number of links connecting the n th node in a walk with nodes already visited coincides, within our approximation, with N'_{k_0} . (Note that there is one such link available for each of those N'_{k_0} nodes.)

The probability of finding a blocking node in step n is given by the ratio $q_n = \sum_k N'_k / N_{\text{end}}$, where $N_{\text{end}} = \langle k \rangle N$ is the total number of ends of links in the network. Then, to order n/N , we have $q_n \approx N'_{k_0} / \langle k \rangle N$. This approximation relies on the fact that $p_n = wn/N \ll 1$, which may be unfulfilled when the minimum degree k_0 is large and the average attrition length can be on the order of the system size N (see below).

To derive the probability distribution for the attrition length L , we consider M_0 kinetic growth walks starting from nodes taken at random. We call $M_2(n)$ the number of walks that survive after n steps. Then, the number of walks finishing at step n (for which n is a blocking one) is $M_2(n) - M_2(n+1) = q_n M_2(n)$. Considering again n as a continuous variable x , one has a differential equation for $M_2(x)$,

$$\frac{1}{M_2} \frac{dM_2}{dx} = -Y \left(\frac{x}{N} \right)^{k_0-1}, \quad (19)$$

with the network-dependent constant

$$Y = \frac{N_{k_0} k_0}{N \langle k \rangle} w^{k_0-1} \quad (20)$$

and w given in Eq. (13). Then, for integer n we have

$$M_2(n) = M_0 \exp \left[- \left(\frac{n}{x_0} \right)^{k_0} \right], \quad (21)$$

which gives the number of walks that remain after n steps; i.e., $M_2(n)/M_0$ is the probability of surviving to length n . Here x_0 is a number (dimensionless length) given by $x_0^{k_0} = k_0 N^{k_0-1} / Y$.

Therefore, the probability distribution $Z(L)$ for the attrition length of these walks is

$$Z(L) = \frac{k_0}{L} \left(\frac{L}{x_0} \right)^{k_0} \exp \left[- \left(\frac{L}{x_0} \right)^{k_0} \right]. \quad (22)$$

This distribution is strongly dependent on the minimum degree k_0 , since nodes with this degree are in fact controlling

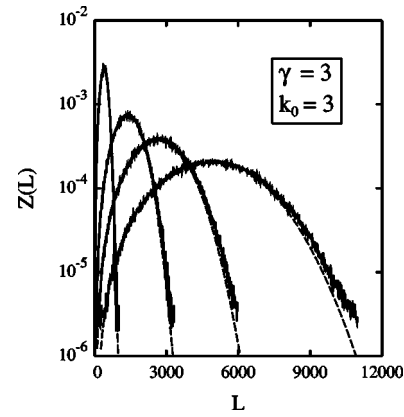


FIG. 6. Distribution probability $Z(L)$ for the attrition length L of kinetic growth walks on scale-free networks with $\gamma=3$, $k_0=3$, and different system sizes. From left to right: $N=3.3 \times 10^3$, 2.6×10^4 , 7.7×10^4 , and 2.1×10^5 . Solid and dashed lines indicate results of numerical simulations and analytical calculations, respectively.

the maximum length of kinetic growth walks in scale-free networks. The distribution $Z(L)$ is displayed in Fig. 6 for $\gamma=3$, $k_0=3$, and different system sizes. The dashed lines were obtained from Eq. (22), and the solid lines were derived from numerical simulations. Both sets of results follow the same trend, but the numerical results seem to be larger than the analytical ones for large L . This difference is larger than the noise of the simulation results and shows the validity limit of our approximation for large L .

From the distribution $Z(L)$ we obtain an average attrition length

$$\langle L \rangle \approx \frac{x_0}{k_0} \Gamma \left(\frac{1}{k_0} \right), \quad (23)$$

Γ being Euler's gamma function. Thus, the dependence of $\langle L \rangle$ on N for large systems is controlled by x_0 . To obtain the asymptotic dependence of x_0 , we note that Y in Eq. (20) scales for $\gamma > 2$ as $Y \sim w^{k_0-1}$, because N_{k_0}/N converges to a constant for large N . In addition, $w \sim \langle k^2 \rangle$, and therefore $x_0^{k_0} \sim (N/\langle k^2 \rangle)^{k_0-1}$. For $\gamma > 3$, $\langle k^2 \rangle$ converges to a finite value as $N \rightarrow \infty$ and the average attrition length increases as $\langle L \rangle \sim N^{1-1/k_0}$. For $\gamma=3$, $\langle k^2 \rangle \sim \ln N$ and $\langle L \rangle \sim (N/\ln N)^{1-1/k_0}$. This means that for a given system size, the average number of nodes visited in kinetic growth walks rises with increasing k_0 , as a consequence of the increase in the average degree $\langle k \rangle$. For $\gamma < 3$, we have $\langle L \rangle \sim N^\alpha$, with an exponent α that decreases from $1 - 1/k_0$ to $(1 - 1/k_0)/2$ as γ decreases from 3 to 2.

In Fig. 7 we show the average attrition length $\langle L \rangle$ as a function of the system size N for $\gamma=3$. Symbols correspond to results of numerical simulations for several values of the minimum degree k_0 , and dashed lines were obtained by using Eq. (23). For the largest k_0 , $\langle L \rangle$ derived from simulations increases with N slightly faster than the analytical result. This difference is not strange if one observes that for $k_0=9$, $\langle L \rangle$ is on the order of N (in fact, for $N=10^5$, $N/\langle L \rangle \approx 3$), and

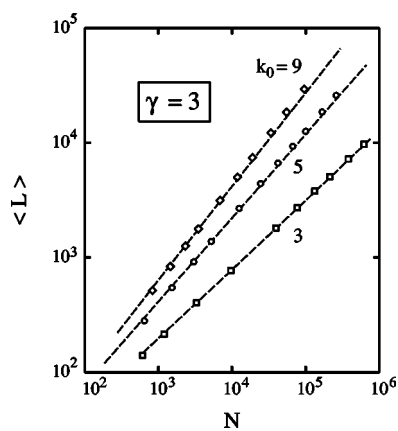


FIG. 7. Mean attrition length $\langle L \rangle$ as a function of system size for SF networks with $\gamma=3$ and several values of k_0 . From top to bottom: $k_0=9, 5$, and 3 . Symbols are results of numerical simulations, and dashed lines were obtained by using Eq. (23). Error bars of simulation results are less than the symbol size.

our assumption that $n \ll N$ for all steps of SAW's is not true. However, even in this case Eq. (23) gives a rather good approximation for the average length $\langle L \rangle$ (see Fig. 7).

In order to define strategies to search in this kind of networks, it is clear that nodes with a low degree limit the effectiveness of the process. Thus, actual strategies based on SAW's have to include additional conditions to improve their efficiency. Along these lines, Adamic *et al.* [28] have proposed an algorithm based on SAW's that prefer high-degree nodes to low-degree ones. In any case, the long-range properties of pure SAW's give us direct insight into the structure of SF networks, further than the local neighborhood of a node, where the structure of links is tree like. On a larger scale, one always finds loops sooner or later in finite networks, which is in fact probed by SAW's. In particular, the average self-intersection length $\langle l \rangle$ given in Eq. (17) is a measure of the typical size of loops in equilibrium SF networks. The presence of loops in a network is responsible for attrition of the walks. Then, the mean attrition length $\langle L \rangle$ given in Eq. (23) is a measure of the long-range "openness" of a network (the longer $\langle L \rangle$, the less loops contain a network). In this sense, our SF networks become more open as their size N increases and eventually are loop-free (or tree like) in the thermodynamic limit, where both $\langle l \rangle$ and $\langle L \rangle$ diverge to infinity.

As a result, we find that the efficiency of SAW's to explore scale-free networks increases for increasing exponent γ . This is a consequence of the fact that for a given system size N , the fraction of nodes with high degree increases for decreasing γ . High-degree nodes are visited more probably than low-degree ones, and once visited, the former are more effective to block a SAW in later steps (they have more connections), thus reducing the mean self-intersection and attrition lengths.

VI. CONCLUSIONS

Self-avoiding walks provide us with an adequate tool to study the long-range characteristics of SF networks. In particular, they allow us to study the quality of a network to be explored without returning to sites already visited. For large networks, the number of SAW's increases as $s_n/s_{n-1} \approx \langle k^2 \rangle / \langle k \rangle - 1$, provided that $n \ll N$. For a given n , s_n decreases with decreasing system size, as a consequence of the presence of loops in the networks. These finite-size effects affect strongly the maximum length of kinetic growth walks on scale-free networks.

We have calculated self-intersection and attrition lengths by using an approximate probabilistic method, which yields results in good agreement with those derived from numerical simulations. Both the average self-intersection length and attrition length scale as a power of the system size N . For the mean self-intersection length of nonreversal SAW's we have $\langle l \rangle \sim N^\beta$, with β depending on the exponent γ of the degree distribution. In particular, for $\gamma > 3$ one has $\beta=0.5$ and decreases as γ is lowered. The length of kinetic growth walks in scale-free networks is limited by attrition of the paths, and the mean attrition length follows a dependence $\langle L \rangle \sim N^\alpha$, with α depending on γ and the minimum degree k_0 . For $\gamma > 3$, one has $\alpha=1-1/k_0$. This dependence of the exponent α on k_0 is remarkable, reflecting the fact that the length of SAW's is limited by attrition at sites with the minimum degree k_0 .

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